

Straight-line string with curvature

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Abstract

Classical and quantum solutions for the relativistic straight-line string with arbitrary dependence on the world surface curvature are obtained. They differ from the case of the usual Nambu-Goto interaction by the behaviour of the Regge trajectory which in general can be nonlinear. Regularization of the action is considered and comparison with relativistic point with curvature is made.

Straight-line string (defined precisely in what follows) is a system with finite number degrees of freedom. Its quantization does not encounter anomalies appropriate to the general string. Nevertheless it is an interesting system to study quantization and various types of interaction. It is a simple extended relativistic object which can be used to build phenomenological hadron models. In this paper we solve the problem of straight-line string when its interaction arbitrarily depends on curvature. We will discuss behaviour of Regge trajectories in different relativistic models and importance of regularization for straight-line string interactions.

Let us consider a straight-line string with the action

$$A = \int_{\tau_1}^{\tau_2} \int_{\sigma_1}^{\sigma_2} F(R/2) \sqrt{-g} d\sigma d\tau, \quad (1)$$

where σ and τ are Poincare invariant space-like and time-like parameters describing the string and its evolution, g is the determinant of the string world surface metric, R is the scalar curvature of the surface and $F(x)$ is an arbitrary function, satisfying a boundary condition. To formulate this condition we mention that the integrand in (1) should vanish at the ends of an open string. This follows from variation of (1) with respect to the ends of the string. We shall assume that the ends of the string are determined by vanishing of the metric, so that F is less singular than $1/\sqrt{-g}$ when $g \rightarrow 0$. We shall call this case a proper string model to distinguish it from a general

model which is obtained from a proper string model by analytic continuation with respect to a parameter in F . For $F(x) = x$ the integrand in (1) is a full derivative.

The straight-line string is described by the vector (for definiteness we consider 4-dimensional Minkowski space with metric $diag(+1, -1, -1, -1)$)

$$x^\mu(\sigma, \tau) = r^\mu(\tau) + q^\mu(\tau)f(\sigma, \tau), \quad (2)$$

where the Poincare vector r^μ corresponds to a point on the string, the translation invariant Lorentz vector q^μ describes the direction of the string and the Poincare invariant function f determines the position of points on the string. r , q and f are the dynamical variables of the straight-line string.

It is shown in the Appendix that for the straight-line string (2) the determinant of the metric and the scalar curvature are given by the simple expressions:

$$g = q^2(\dot{r}_q + \dot{q}_q f)^2 f'^2, \quad (3)$$

$$R/2 = -\frac{\dot{r}_q^2 \dot{q}_q^2 - (\dot{r}_q \dot{q}_q)^2}{q^2((\dot{r}_q + \dot{q}_q f)^2)^2}, \quad (4)$$

where dot and prime mean derivatives with respect to τ and σ , respectively, and index q means that the corresponding vector is orthogonal to q :

$$z_q = z - (zq)q/q^2, \quad z = \dot{r}, \dot{q}. \quad (5)$$

Putting (3) and (4) into (1) we can derive the Euler-Lagrange equations for r , q and f . The equations for r and q contain integrals over $d\sigma$ and equation for f is identically satisfied. Therefore, we prefer to integrate over $d\sigma$ in (1) at the beginning. The integration reduces to the integration over df in the limits which correspond to zeros of g . As a result we have the action and the Lagrangian of our model:

$$A = \int_{\tau_1}^{\tau_2} L d\tau, \quad (6)$$

$$L = (-\dot{n}^2)^{1/2} G(l), \quad (7)$$

where n is the unit vector in the q -direction:

$$n = q/(-q^2)^{1/2}, \quad (8)$$

l is the radius (half of length) of the string at fixed τ

$$l = (q^2(\dot{r}_q^2 \dot{q}_q^2 - (\dot{r}_q \dot{q}_q)^2)/(\dot{q}_q^2)^2)^{1/2} \quad (9)$$

and the function G is obtained from F in (1) by the integration

$$G(l) = l^2 \int_0^1 F((lx)^{-2})(x/(1-x))^{1/2} dx. \quad (10)$$

By the assumption this integral exists or is defined through an analytic continuation.

To find an extremum of the action (6) and (7) we shall use the Hamilton method. The momenta canonically conjugate to the "external" and "internal" coordinates r and q are

$$p_\mu = -\partial L / \partial \dot{r}^\mu, \quad \pi_\mu = -\partial L / \partial \dot{q}^\mu. \quad (11)$$

Their non-zero Poisson brackets are

$$\{r^\mu, p_\nu\} = \{q^\mu, \pi_\nu\} = -\delta_\nu^\mu. \quad (12)$$

Because of the transformation properties of the action, r and q with respect to the Poincare group, the total conserved (independent of τ) energy-momentum vector of the system coincides with p and the conserved tensor of the total angular momentum is given by

$$M^{\mu\nu} = r^\mu p^\nu - r^\nu p^\mu + q^\mu \pi^\nu - q^\nu \pi^\mu. \quad (13)$$

The conserved spin tensor is

$$S^{\mu\nu} = M^{\mu\nu} - (M^{\mu\rho} p^\nu + p^\mu M^{\rho\nu}) p_\rho / p^2. \quad (14)$$

For the mass and spin of our system we shall use the notations

$$m = \sqrt{p^2}, \quad S = \sqrt{S^{\mu\nu} S_{\mu\nu} / 2}. \quad (15)$$

The action (6)-(9) is invariant under three sets of τ -dependent transformations: shift of r in the direction of q , change of q^2 and reparametrization

of τ . Therefore there should be three constraints on the canonical variables. Calculating momenta (11) from (7)-(9) we get

$$pq = 0, \quad \pi q = 0, \quad (16)$$

$$S = K(m). \quad (17)$$

Here, with constraints (16) fulfilled,

$$S^2 = q^2(\pi^2 - (\pi p)^2/p^2) \quad (18)$$

and the function K is determined by the function G by means of the parametric equations

$$m = |G'(l)|, \quad (19)$$

$$S = |G(l) - lG'(l)|, \quad (20)$$

where prime stands for the derivative with respect to l .

Introducing the constraint functions (constraints)

$$\varphi_1 = pq, \quad \varphi_2 = \pi q, \quad (21)$$

$$\varphi_3 = S - K(m), \quad (22)$$

we see that they are constraints of the first kind, i.e. their Poisson brackets vanish when the constraints are equal to zero. The canonical Hamiltonian of our system $H_{can} = -p\dot{r} - \pi\dot{q} - L = 0$, therefore the dynamics of our system is determined by the Hamiltonian

$$H = \sum_{i=1}^3 v_i(\tau) \varphi_i \quad (23)$$

and the equation

$$\dot{f} = \partial f / \partial t + \{f, H\}, \quad (24)$$

where v_i is arbitrary and f is a function of the canonical variables.

For the canonical variables themselves equations (24) can be easily solved:

$$r = 2KK'pt_3/m - qt_1, \quad (25)$$

$$q = e^{-t_2}(q_0 \cos T + \pi_p(q_0^2/\pi_p^2)^{1/2} \sin T), \quad (26)$$

$$\pi = e^{t_2}(\pi_p \cos T - q_0(\pi_p^2/q_0^2)^{1/2} \sin T) + pt_1. \quad (27)$$

Here $T = 2Kt_3 + v_0$, $K = K(m)$ is given by (17),(19),(20), $K' = dK/dm$, and v_0 is a constant. q_0 and π_p are constant vectors satisfying (16), π_p is orthogonal to p , $\pi_p p = 0$, and they are connected with spin

$$S = (q_0^2 \pi_p^2)^{1/2}. \quad (28)$$

t_i , $i = 1, 2, 3$, are arbitrary functions of τ :

$$t_1 = e^{t_2}(c_1 + \int v_1 e^{-t_2} d\tau), \quad t_j = c_j + \int v_j d\tau, \quad j = 2, 3. \quad (29)$$

It is not difficult to show that the first term on the r.h.s. of (25) is the coordinate of the center of the string. Equating its time component to the laboratory time

$$t = 2KK'p^0 t_3/m, \quad T = mt/K'p^0 + v_0, \quad (30)$$

we see that in the laboratory frame the center of the string is moving with constant velocity \vec{p}/p^0 and the direction of the string is rotating in the plane orthogonal to p^μ and $S^\mu = \epsilon_{\mu\nu\rho\sigma} p^\nu M^{\rho\sigma}/2m = \epsilon_{\mu\nu\rho\sigma} p^\nu q_0^\rho \pi_p^\sigma/m$ with the angular velocity

$$\omega = m/K'p^0 = (m/p^0)(dm/dS) = (m/p^0)l^{-1}. \quad (31)$$

To quantize our system we use the gauge conditions

$$p\pi = 0, \quad q^2 = \pi^2 \quad (32)$$

and the tetrad of orthonormal vectors e_α , $\alpha = 0, a$, $a = 1, 2, 3$

$$e_0 = p/\sqrt{p^2}, \quad e_\alpha e_\beta = g_{\alpha\beta}. \quad (33)$$

Solving the constraint and gauge conditions (16) and (32) we get

$$q = e_a n^a (\vec{S}^2)^{1/4}, \quad \pi = e_a [\vec{S}, \vec{n}]^a (\vec{S}^2)^{-1/4}, \quad (34)$$

$$S^{\mu\nu} = e_a^\mu e_b^\nu \epsilon_{abc} S^c, \quad S^2 = \vec{S}^2, \quad (35)$$

$$\vec{n}^2 = 1, \quad \vec{n}\vec{S} = 0. \quad (36)$$

To obtain the Poisson (Dirac) brackets of the variables \vec{n}, \vec{S}, r and p one can use method of reduction of the symplectic form [1,2,3]. Using the initial symplectic form

$$\omega = dp_\nu \wedge dr^\nu + d\pi_\nu \wedge dq^\nu \quad (37)$$

and equations (34) we get

$$d\pi_\nu \wedge dq^\nu = dp_\nu \wedge du^\nu - d[\vec{S}, \vec{n}] \wedge d\vec{n}, \quad (38)$$

$$u^\nu = (1/2)\epsilon_{abc}e_{a\mu}(\partial e_b^\mu/\partial p_\nu)S^c. \quad (39)$$

Therefore,

$$\omega = dp_\nu \wedge dz^\nu - d[\vec{S}, \vec{n}] \wedge d\vec{n}, \quad (40)$$

where

$$z^\nu = r^\nu + u^\nu. \quad (41)$$

Taking into account (36) and calculating the inverse matrix to that in (40) we get

$$\{z^\mu, p_\nu\} = -\delta_\nu^\mu, \quad \{S^a, S^b\} = -\epsilon_{abc}S^c, \quad \{S^a, n^b\} = -\epsilon_{abc}n^c \quad (42)$$

Using these Poisson brackets it is easy to see that the Poisson brackets of the energy-momentum vector p^μ and the angular momentum tensor $M^{\mu\nu}$ (13),(34) form a representation of the algebra of the Poincare group.

Now it is easy to quantize our system. We replace the variables z, p, \vec{n}, \vec{S} by operators satisfying (36) and the Poisson brackets in (42) by commutators $\{, \} \rightarrow -i[,]$. The wave function of a physical state satisfies the equation

$$\hat{\varphi}_3\psi \equiv (\sqrt{\hat{\vec{S}}^2} - K(\sqrt{\hat{p}^2}))\psi = 0. \quad (43)$$

In the representation where \hat{p} and $\hat{\vec{n}}$ are diagonal a basis of physical states is given by

$$\psi_{\vec{k}SS^3}(p, \vec{n}) = c\delta(\vec{p} - \vec{k})\delta(p^0 - \sqrt{\vec{k}^2 + m_S^2})Y_{S^3}^S(\vec{n}), \quad (44)$$

where $Y_{S^3}^S(\vec{n})$ is the spin eigenfunction,

$$\sqrt{S(S+1)} = K(m_S) \quad (45)$$

and S is a non-negative integer. It is important to notice that the operator of the angular momentum obtained from (13),(34),(39),(41) does not contain

noncommuting multipliers. Therefore, the commutators of $\hat{M}^{\mu\nu}$ and \hat{p}^ρ coincide (up to i) with their Poisson brackets and form a representation of the algebra of the Poincare group, i.e. our quantized theory is Poincare invariant.

Let us discuss the obtained results. First we mention the problem of regularization of integrals (1) and (10). Even in a trivial case $F(x) = x$, when Lagrangian (1) is a full derivative, we encounter divergence in integral (10). The boundary conditions at the ends of the string are not fulfilled either. We need a regularization, for instance $F(x) = x^\gamma$ to get (52), from where we get zero action at $\gamma \rightarrow 1$, as it should be. Let us consider from this point of view the Polyakov interaction[5] in the straight-line approximation. Polyakov suggested a term in the string action proportional to

$$\int \int P \sqrt{-g} d\sigma d\tau, \quad P = \sum_{a=1}^2 (Spb^{(a)})^2, \quad (46)$$

where $b^{(a)}$ is the second quadratic form of the string world surface

$$b_{ik}^{(a)} = (\partial^2 x / \partial \sigma^i \partial \sigma^k, n^{(a)}), \quad \sigma^1 = \sigma, \quad \sigma^2 = \tau, \quad (47)$$

$n^{(a)}, a = 1, 2$ being the orthonormal vectors orthogonal to the surface and $Spb^{(a)} = g^{ik} b_{ki}^{(a)}$. In the conformal gauge

$$P = (\ddot{x} - x'')^2 / g. \quad (48)$$

In the straight-line approximation (2)

$$P = -\frac{(a + bf)^2}{[(\dot{r}_q + \dot{q}_q f)^2]^2}, \quad a = \ddot{r}_q - 2\dot{q}_q(\dot{r}_q)/q^2, \quad b = a(r \rightarrow q) \quad (49)$$

(see Appendix) and the integral over $d\sigma(df)$ in (46) diverges. Regularizing (49) by replacing its denominator by its γ th power (it is not a gauge invariant regularization, but the limit at $\gamma \rightarrow 1$ is gauge invariant) we can integrate over df . Going to the limit $\gamma = 1$ we get a finite Lagrangian

$$L = -\pi(q^2(\dot{q}_q^2)^{-3})^{1/2} b^2, \quad (50)$$

which does not depend on the derivatives of r . This result is not satisfactory because it corresponds to zero energy-momentum vector $p = 0$. Therefore

the straight-line approximation (or the aforementioned regularization) is not valid for this particular interaction.

Next we see that our relativistic model (7) possesses a Regge trajectory, i.e. spin of the system depends on its mass and not on other continuous dynamical variables. The origin of this phenomenon is the high symmetry of the Lagrangian (7): it is invariant not only under reparametrization, but also under a shift of r in the q -direction and under renormalization of q . We can compare our model with the relativistic particle with curvature [4] $L = f(k)(\dot{x}^2)^{1/2}$, $k = (-\ddot{x}^2)^{1/2}/\dot{x}^2$. Here \dot{x} can be treated as a new coordinate, so this system has the same number of degrees of freedom as (7). But because of absence of the aforementioned symmetry and corresponding constraints spin of this system depends not only on its mass, but also on two canonically conjugate variables which vary from $-\infty$ to $+\infty$. Only for $f(k) = k$ there exists an additional constraint and spin depends on mass only, decreasing with the increase of mass.

In our model, depending on the function F in (1), the behaviour of Regge trajectory may be nonlinear. It is well known that the linear with respect to m^2 rise of Regge trajectories of hadrons composed with the same quarks is a remarkable experimental fact lacking complete theoretical understanding. The straight-line string with the Nambu-Goto interaction $F(x) = \text{const}$ reproduces this behaviour, while many other relativistic models give decreasing trajectories. We see now that in a general relativistic model (7) the behaviour of the Regge trajectory may be nonlinear depending on the function G or F . To see this and the limitations following from the string origin of (7) let us consider an example

$$F(x) = -cx^\gamma. \quad (51)$$

For the proper string model satisfying boundary conditions it is necessary that $\gamma \leq 1/4$. On the other hand, the integrals in (1) and (10) exist if $\gamma < 3/4$ and can be defined by the analytic continuation in γ for $\gamma \neq (2n+3)/4$, $n = 0, 1, 2, \dots$. The function $G(l)$ in (10) is now

$$G(l) = -c\pi^{1/2} \frac{\Gamma(3/2 - 2\gamma)}{\Gamma(2 - 2\gamma)} l^{2-2\gamma} = -bl^{2-2\gamma}. \quad (52)$$

For the total energy of the system to be positive it is necessary that $b(1-\gamma) > 0$. The Regge trajectory (45) is given by

$$\sqrt{S(S+1)} = \beta(m^2)^a, \quad (53)$$

$$a = \frac{1 - \gamma}{1 - 2\gamma}, \quad (54)$$

$$\beta = |b(1 - 2\gamma)|(4b(1 - \gamma))^{-2a}. \quad (55)$$

We see that for the proper string model $1/2 < a \leq 3/2$. For a general relativistic model of type (51) ($\gamma > 1/4$) the Regge trajectory can increase or decrease with $a \neq 0$ and $a \neq (2n - 1)/(4n + 2)$, $n = 0, 1, 2, \dots$. For $F(x) = c + bx^{-A}$, where $A > 0$, spin S (or $\sqrt{S(S + 1)}$ in quantum case) is proportional to m^2 for small mass and to m^a , $a = (A + 1)/(A + 1/2)$ for large mass.

It is easy to see that for $m \rightarrow \infty$ spin is proportional to m^a , $a \geq 1$ for any choice of $G(l)$. We can not have $S \rightarrow \text{const}$ in this limit in our model which is a model of "permanent confinement".

We see that the linear rise of Regge trajectories corresponds to a specific quark-gluon dynamics. On general grounds one can expect deviations from this behaviour, therefore further experimental study of Regge trajectories at higher masses and spins is of considerable theoretical interest. The model considered can serve for a phenomenological description of experimental trajectories.

1 Appendix. Calculation of g , R and A .

Calculation of $g = \det \| (\partial_i x \partial_j x) \|$, $\partial_i = \partial / \partial \sigma^i$ for the straight-line ansatz (2) is straightforward with the result (3) in the text. We can use of course the conformal gauge $\dot{x}x' = 0$, $\rho = \dot{x}^2 = -x'^2$ in which $\dot{f} = -(\dot{r}q + \dot{q}qf)/q^2$ and $-q^2 f'^2 = (\dot{r}q + \dot{q}qf)^2$, so that $g = -\rho^2 = -(q^2 f'^2)^2 = q^2 f'^2 (\dot{r}q + \dot{q}qf)^2$, because we need f' to integrate over df . The use of the conformal gauge is very helpful for calculating R . In this gauge

$$R = (1/\rho)[(\dot{\rho}/\rho)' - (\rho'/\rho)']. \quad (56)$$

Differentiating \dot{f} with respect to σ we get $\dot{f}' = -\dot{q}qf'/q^2$, so that $\dot{\rho} = (-q^2 f'^2)' = 0$. Now $f'' = -(\dot{r}q\dot{q} + \dot{q}_q^2 f)/q^2$, $f''' = -\dot{q}_q^2 f'/q^2$, so that $R = [(f'^2)'/f'^2]/q^2 f'^2 = 2(f'''f' - f''^2)/q^2 f'^4$ and we get the equation (4) in the text.

The same is applied to A in (52). All the derivatives of f can be expressed through f and we get (53) in the text.

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